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THE COEFFICIENT SPACE APPROACH TO THE STABILITY OF MULTIDIMENSIONAL DIGITAL FILTERS

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ABSTRACT

This report is concerned with the development of a new approach to the problem of stability for multidimensional, causal, recursive, 'all pole', digital filters. The distinguishing feature of this approach is that general stability criteria can be derived directly in terms of the coefficients of the transfer function of the filter. Thus by use of this method it is sometimes possible to determine which coefficients of the transfer function are critical to the stability of the filter, information which is, of course, important in filter design. Also the emphasis of this approach is on the development of a conceptual method for considering the problem in complete generality.

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This report is concerned with the development of a new approach to the problem of stability for multidimensional, causal, recursive, 'all pole', digital filters. The distinguishing feature of this approach is that general stability criteria can be derived directly in terms of the coefficients of the transfer function of the filter. Thus by use of this method it is sometimes possible to determine which coefficients of the transfer function are critical to the stability of the filter, information which is, of course, important in filter

design. Also the emphasis of this approach is on the development of a conceptual method for considering the problem in complete generality.

The Coefficient Space Approach to the Stability of Multidimensional Digital Filters

1. Introduction

Consider a filter whose transfer function has the form

$$H(\vec{z}) = 1/(1 - Q(\vec{z}^{-1})) \tag{1.1}$$

where $\vec{z} = (z_1, \ldots, z_N)$ is a vector complex variable, and $Q(\vec{z})$ is a polynomial in N variables with real coefficients and zero constant term. Such a transfer function describes an 'all-pole' 'causal', recursive, multidimensional digital filter, and it is known that questions of stability generally reduce to a question of stability for filters of this type. It is also known that such a filter is stable iff

$$P(\vec{z}) = 1 - Q(\vec{z}) \tag{1.2}$$

has no zeros $\vec{\beta} = (\beta_1, \ldots, \beta_N)$ such that $|\beta_i| \leq 1$ for all i. For this reason, it is convenient terminology to state that a polynomial $P(\vec{z})$ of several variables is <u>stable</u> if $P(\vec{\beta}) = 0$ implies that $|\beta_i| > 1$ for some i. The question of stability then becomes, primarily, the problem of determining if such a polynomial is stable.

An N-tuple of the form $\vec{j} = (j_1, \ldots, j_N)$, where each j_i is a non-negative integer is called a <u>vector index</u>. If for such an index $\vec{j}$, we define $\vec{z}^{\vec{j}}$ by

$$\vec{z}^{\vec{j}} = z_1^{j_1} z_2^{j_2} \ldots z_N^{j_N} \tag{1.3}$$

then every polynomial $P(\vec{z})$ in N-variables can be written in the form

$$P(\vec{z}) = \sum_{\vec{j}} a_{\vec{j}} \vec{z}^{\vec{j}} \tag{1.4}$$

where $a_{\vec{j}}$ for each $\vec{j}$, is a real number, equal to zero except for a finite number of indices (see Davis and Souchon [1975]). For example if

$$P(z_1, z_2) = 1.00 + 0.5z_1z_2 - .11z_1^2z_2$$

then $a_{00} = 1.00$, $a_{11} = .05$, $a_{21} = -.11$ and all other coefficients equal zero.

Suppose that a fixed, ordered set $(\vec{j}_1, \vec{j}_2, \ldots, \vec{j}_K)$ of non-zero, N-dimensional vector indices is given, and consider the set of polynomials of the form

$$P(\vec{z}) = 1 - \sum_{i=1}^{K} a_{\vec{j}_i} \vec{z}^{\vec{j}_i} \tag{1.5}$$

where, for each i , $a_{\vec{j}_i}$ is a real number. Each such polynomial (1.5) is naturally associated to the point $(a_{\vec{j}_1}, \ldots, a_{\vec{j}_K})$ of K-dimensional euclidean space. Conversely, each point $(A_1, \ldots, A_K)$ of K-dimensional euclidean space is associated to a polynomial of type (1.5) by the correspondence

$$A_i = a_{\vec{j}_i} , \quad i = 1, \ldots, K \tag{1.6}$$

relative to the ordered set of indices $\vec{j}_i, \ldots, \vec{j}_K$. Thus relative to this ordered set of indices, K-dimensional space becomes the <u>coefficient space</u> for polynomials of type (1.5). A point $(A_1, \ldots, A_K)$ of this coefficient space will be said to be a <u>stable point</u> if its associated polynomial, namely

$$P(\vec{z}) = 1 - \sum_{i=1}^{K} A_i \, \vec{z}^{\,\vec{J}_i} \tag{1.7}$$

is stable. The set of all such stable points in the coefficient space will be called the <u>region of stability</u>. The problem that will be considered here is what can be found concerning the region of stability for a given set of vector indices, or equivalently, for a given type of polynomial.

In certain cases, the region of stability can be specified completely. Consider the ordered set of indices $\vec{J}_1 = (10)$, $\vec{J}_2 = (01)$ $\vec{J}_3 = (11)$. The associated polynomials are of the form

$$P(z_1,z_2) = 1 - A_1 z_1 - A_2 z_2 - A_3 z_1 z_2 \,, \tag{1.8}$$

and the associated coefficient space is 3 dimensional euclidean space. Huang [1972] has shown that (1.8) is stable iff

$$|A_3| < 1$$

$$|A_1 + A_2| < 1 - A_3$$

$$|A_1 - A_2| < 1 + A_3 \quad .$$

The region of stability is illustrated in Figure 1. Also, it can be shown, see Jury [1974] that a similar type of specification can be given for any region of stability associated to polynomials in one variable, or equivalently, to any ordered set of 1-dimensional vector indices.

Although, regions of stability can be very difficult to determine for more complicated types of multivariable polynomials, in the following it will be shown that several properties of regions of stability can be derived in general.

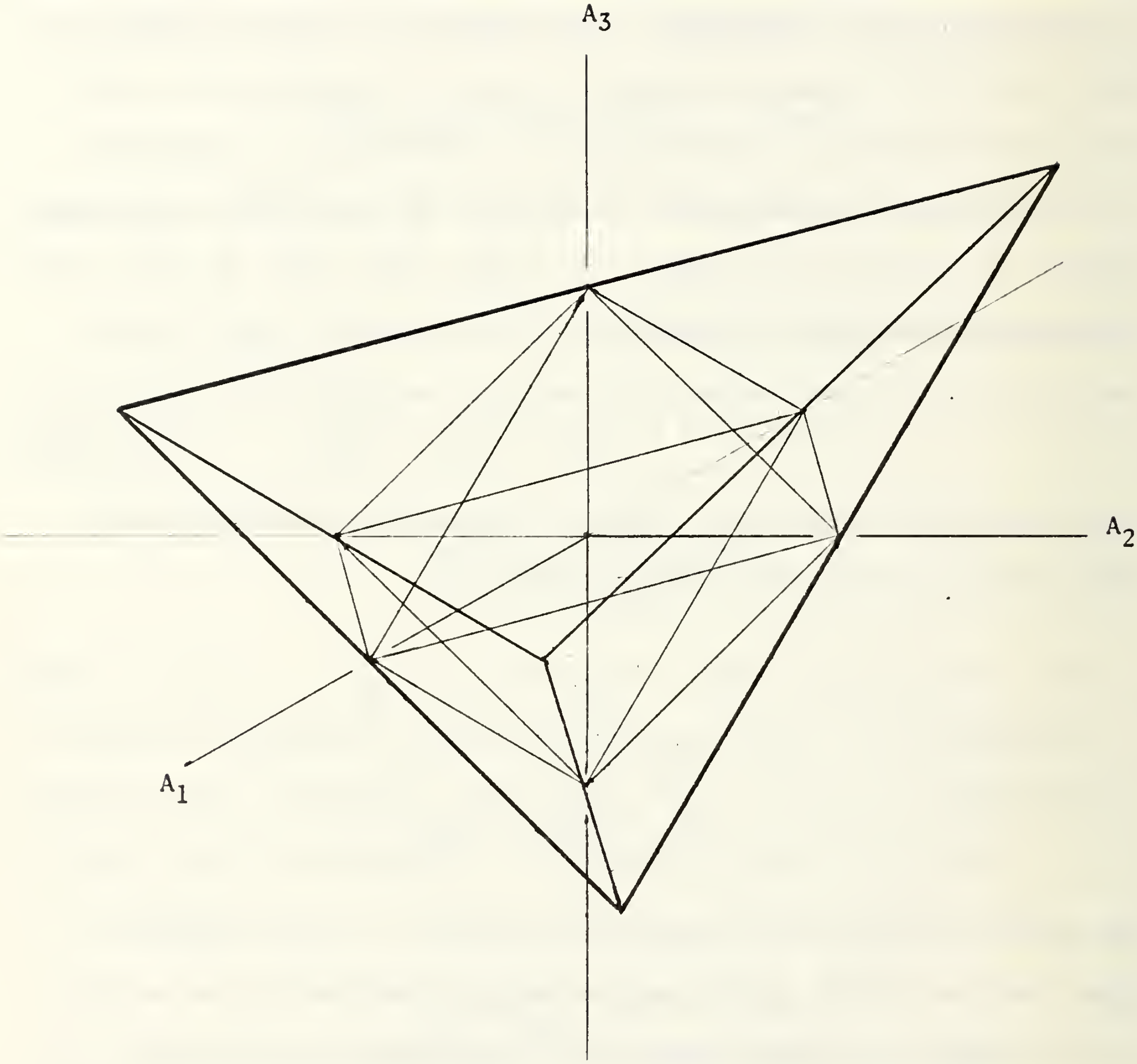

FIGURE 1

2. The Basic Theorems

Throughout the following it will be assumed that $\vec{j}_1, \ldots, \vec{j}_K$ is a given, fixed, ordered set of N dimensional vector indices. Thus the correspondence between polynomials of the form (1.7) and points $(A_1, \ldots, A_K)$ of K-dimensional space is assumed fixed.

The <u>order</u> of a vector index $\vec{j} = (j_1, \ldots, j_N)$ is defined by

$$\text{order } \vec{j} = j_1 + \ldots + j_N .$$

A coefficient A_i of a polynomial of type (1.7) is called a <u>leading coefficient</u> if its associated vector index $\vec{j}_i$ has maximal order. A polynomial of several variables may have several leading coefficients.

<u>Theorem 2.1</u> If $(A_1, \ldots, A_K)$ is a stable point of the coefficient space and if A is the sum of the leading coefficients of the polynomial corresponding to this point, then $|A| < 1$.

Proof. Let $P(\vec{z})$ be the stable polynomial associated to $(A_1, \ldots, A_K)$. It follows that the one variable polynomial $p(z)$ defined by

$$p(z) = P(z, \ldots, z) \tag{2.1}$$

is also stable. Moreover the leading coefficient of $p(z)$ is seen to be A. The polynomial $p(z)$ factors,

$$p(z) = A(z-t_1)\ldots(z-t_M) \tag{2.2}$$

where $t_1, \ldots, t_M$ are its complex roots. Since $p(z)$ is stable it follows that

$$|t_i| > 1 \qquad i = 1, \ldots, M . \tag{2.3}$$

But then,

$$1 = |p(0)| = |A| \left| \prod_{i=1}^{M} t_i \right| \tag{2.4}$$

where

$$\left| \prod_{i=1}^{M} t_i \right| > 1 . \tag{2.5}$$

Hence,

$$|A| < 1 . \tag{2.6}$$

<u>Theorem 2.2</u> (Necessity Theorem) Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$ be any N-tuple of complex numbers such that for each $i = 1, \ldots, N$, $|\alpha_i| = 1$, and for each i

$$s_i = \vec{\alpha}^{\vec{J}_i} \tag{2.7}$$

is a <u>real</u> number. Then, if $(A_1, \ldots, A_K)$ is any stable point, it must be true that

$$s_1 A_1 + \ldots + s_K A_K < 1 . \tag{2.8}$$

(Note that since each s_i has modulus 1 and is real, each s_i must equal +1 or -1.)

Proof. Let $P(\vec{z})$ be the stable polynomial associated to $(A_1, \ldots, A_K)$. Suppose $\vec{\alpha}$ satisfies the hypotheses and define $p(z)$ by

$$p(z) = P(\alpha_1 z, \ldots, \alpha_N z) . \tag{2.9}$$

Then $p(z)$ is a one variable polynomial with <u>real</u> coefficients. Moreover

$$p(1) = 1 - \sum_{i=1}^{K} A_i(\alpha_1, \ldots, \alpha_N)^{\vec{J}_i}$$
$$= 1 - \sum_{i=1}^{K} A_i s_i \; . \tag{2.10}$$

If

$$\sum_{i=1}^{K} s_i A_i \geq 1 \tag{2.11}$$

then

$$p(1) \leq 0 \; . \tag{2.12}$$

But $p(0) = 1$, and $p(z)$, as a function of a real variable, is clearly continuous. Therefore by the intermediate value theorem for real functions, there must exist a real number t, $0 < t \leq 1$, such that $p(t) = 0$. But then

$$P(\alpha_1 t, \ldots, \alpha_N t) = 0 \tag{2.13}$$

and for each $i = 1, \ldots, N$, $|\alpha_i t| \leq 1$, a contradiction to the assumption that $P(\vec{z})$ is stable. Therefore it must be true that

$$\sum s_i A_i < 1 \; . \tag{2.14}$$

<u>Theorem 2.3</u> (Symmetry Theorem) Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$ be any N-tuple of complex numbers such that for each $i = 1, \ldots, N$, $|\alpha_i| = 1$, and

$$s_i = \vec{\alpha}^{\vec{J}_i} \tag{2.15}$$

is a real number. Then a point $(A_1, \ldots, A_K)$ is stable iff the point $(s_1A_1, \ldots, s_KA_K)$ is stable. (Note as before that each s_i equals +1 or -1.)

Proof. Let $P(z_1, \ldots, z_N)$ be the polynomial associated to $(A_1, \ldots, A_K)$. Define $P'(z_1, \ldots, z_N)$ by

$$P'(z_1, \ldots, z_N) = P(\alpha_1 z_1, \ldots, \alpha_N z_N) . \tag{2.16}$$

It is not difficult to check that P' is the polynomial associated to $(s_1A_1, \ldots, s_KA_K)$. Moreover P' is stable iff P is. For $P'(\beta_1, \ldots, \beta_N) = 0$ iff $P(\beta_1', \ldots, \beta_N') = 0$ where $\beta_i' = \alpha_i\beta_i$ and for each i $|\beta_i'| = |\alpha_i\beta_i| = |\beta_i|$, since $|\alpha_i| = 1$.

Theorem 2.4 (Sufficiency Theorem) Let $(A_1, \ldots, A_K)$ be a point such that

$$\sum_{i=1}^{K} |A_i| < 1 . \tag{2.17}$$

Then $(A_1, \ldots, A_K)$ is stable.

Proof. Consider the polynomial (1.7)

$$P(z_1, \ldots, z_N) = 1 - \sum A_i \, z^{j_i} . \tag{2.18}$$

If $P(\vec{\beta}) = 0$ where $|\beta_i| \leq 1$, then

$$\sum A_i \, \vec{\beta}^{\vec{j}_i} = 1 . \tag{2.19}$$

Therefore

$$1 = \left|\sum_{i=1}^{K} A_i \, \vec{\beta}^{\vec{j}_i}\right| \leq \sum_{i=1}^{K} |A_i| < 1 \tag{2.20}$$

a contradiction.

<u>Corollary 2.1</u> Let $(A_1, \ldots, A_K)$ be a point such that $A_i \geq 0$ for all i. Then $(A_1, \ldots, A_K)$ is stable iff

$$\sum_{i=1}^{K} A_i < 1 . \tag{2.21}$$

Proof. Apply Theorem 2.2 and Theorem 2.4.

Theorem 2.4 and its corollary can be given simple geometric interpretations. The region of points $(A_1, \ldots, A_K)$ satisfying

$$|A_1| + \ldots + |A_K| < 1 \tag{2.22}$$

describe the K-dimensional 'diamond', centered at the origin and whose points lie along the axes. Theorem 2.4 states that the K-dimensional diamond is wholly enclosed by the region of stability. The Corollary states that in the positive 'quadrant', the region of stability always coincides with the 'diamond' (see Figure 1). In the following, the Symmetry Theorem will be applied to show that regions of stability also satisfy certain geometric symmetries.

3. Symmetry

As the proof of Theorem 2.3 shows, symmetries, (in this case sequences of reflections) between the points of the region of stability arise from transformations of the variables of the associated polynomials. The transformation of variable is given by

$$(z_1, \ldots, z_N) \to (z_1', \ldots, z_N') = (\alpha_1 z_1, \ldots, \alpha_N z_N) \tag{3.1}$$

where $(\alpha_1, \ldots, \alpha_N)$ is a complex vector satisfying the conditions that for each i, $|\alpha_i| = 1$ and

$$s_i = \vec{\alpha}^{\vec{j}_i} \tag{3.2}$$

is a <u>real</u> number. Such a stability invariant transformation transforms the coefficients $(A_1, \ldots, A_K)$ into $(s_1A_1, \ldots, s_KA_K)$. Since each s_i always equals +1 or -1, this transformation is geometrically interpreted as a sequence of reflections of the axes.

At least 2^N vectors $\vec{\alpha}$ can be obtained by choosing each α_i equal to +1 or -1. The symmetries of the coefficient space thus obtained, however, are not necessarily distinct, and may not include symmetries which can be obtained by allowing the α_i to be complex. However, symmetries so obtained are more easily studied and for this reason will be called <u>simple</u> symmetries. In the following we will restrict our attention to the study of simple symmetries.

The transformation (symmetry) of the coefficient space determined by the vector $\vec{\alpha}$ can be described completely by the vector $\vec{s} = (s_1, \ldots, s_K)$ specified by eqn. (3.2). Each coordinate of the vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$ and $\vec{s} = (s_1, \ldots, s_K)$ equals +1 or -1. However for reasons which will become clear, it will be more convenient to use 1 in place of -1, and 0 in place of 1. The multiplicative relation (eqn. 3.2) between the vectors $\vec{s}$ and $\vec{\alpha}$ now becomes the following additive one in modulo 2 arithmetic.

$$s_i = \vec{\alpha} \cdot \vec{j}_i \pmod 2 \tag{3.3}$$

where the '$\cdot$' represents the vector dot product of the 0, 1 vector $\vec{\alpha}$ with the integer vector index $\vec{j}_i$. If the $N \times K$ integer matrix J is defined by

$$J = (\vec{j}_1^{\,T}, \ldots, \vec{j}_K^{\,T}) \tag{3.4}$$

then the vector $\vec{s}$ which describes the simple symmetry arising from the change of variables described by the vector $\vec{\alpha}$ satisfies

$$\vec{s} = \vec{\alpha}\, J \pmod 2. \tag{3.5}$$

Moreover the 2^N possible choices for $\vec{\alpha}$ can now be viewed as the elements of the N-dimensional vector space over the Galois field, GF(2), of two elements, a tool familiar in algebraic coding theory (Berlekamp [1968]). The simple symmetries can now be easily classified using the linear algebra of these vector spaces.

<u>Theorem 3.1</u> The set of K-dimensional vectors over GF(2) which describe the set of simple symmetries of the region of stability is the set of vectors spanned, modulo 2, by the row vectors of the matrix J .

Proof. Each vector $\vec{s}$ arises by eqn. 3.5 from an $\vec{\alpha}$. Each $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$ can be written as

$$\vec{\alpha} = \alpha_1\vec{e}_1 + \ldots + \alpha_N\vec{e}_N \tag{3.6}$$

where

$$\vec{e}_i = (0, \ldots, 1, \ldots, 0) \tag{3.7}$$

with a 1 in the i^{th} coordinate. But then

$$\vec{s} = \alpha_1(\vec{e}_1 J) + \ldots + \alpha_N(\vec{e}_N J)$$

by linearity; and for each i , $\vec{e}_i J$ is the i^{th} row of J .

(Note also that by the above theorem the set of simple symmetries forms an abelian group.)

Corollary 3.1 The number of simple symmetries equals 2^L, where L is the modulo 2 rank of the matrix J.

Proof. Immediate.

Example 3.1

Consider the filter (eqn. 1.8) studied by Huang. The vector indices in this case are

$$\begin{aligned} \vec{J}_1 &= (10) \\ \vec{J}_2 &= (01) \\ \vec{J}_3 &= (11) \quad . \end{aligned} \tag{3.8}$$

The matrix J is therefore

$$J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad . \tag{3.9}$$

The modulo 2 rank of J is clearly 2. Thus there are 4 $(=2^2)$ simple symmetries. Each symmetry is described by an element of the row space of J. They are:

$$\begin{aligned} \vec{s}_1 &= (000) \\ \vec{s}_2 &= (101) \\ \vec{s}_3 &= (011) \\ \vec{s}_4 &= (110) \end{aligned} \tag{3.10}$$

which describes the symmetries

$$\begin{aligned} (A_1,A_2,A_3) &\rightarrow (A_1,A_2,A_3) \\ &\rightarrow (-A_1,A_2,-A_3) \\ &\rightarrow (A_1,-A_2,-A_3) \\ &\rightarrow (-A_1,-A_2,A_3) \quad . \end{aligned} \tag{3.11}$$

Note that in Figure 1, there are only two different basic shapes for the region of stability in the eight different quadrants of the coefficient space, and that consistent with the above symmetries each shape occurs symmetrically in four quadrants.

<u>Example 3.2</u>

Consider a filter whose transfer function is

$$1/P(z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}) \tag{3.12}$$

where

$$P(z_1,z_2,z_3,z_4) = 1 - A_1 z_1 z_2^3 - A_2 z_1 z_2^2 z_3^2 z_4 - A_3 z_1 z_2^3 z_3^3 z_4 \tag{3.13}$$

$$- A_4 z_2^3 z_3^3 z_4 - A_5 z_1 z_3^3 z_4 \; .$$

In this case the vector indices are

$$\begin{aligned} \vec{J}_1 &= (1301) \\ \vec{J}_2 &= (1221) \\ \vec{J}_3 &= (1331) \\ \vec{J}_4 &= (0331) \\ \vec{J}_5 &= (1031) \; . \end{aligned} \tag{3.14}$$

And the matrix J is therefore

$$J = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 3 & 2 & 3 & 3 & 0 \\ 0 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \; . \tag{3.15}$$

Calculating modulo 2, and row reducing,

$$J \equiv \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3.16}$$

Therefore J has rank 4, and there are $2^4 = 16$ possible distinct simple symmetries. Moreover each possible symmetry can be described by a modulo 2 sum of the rows of J. For example adding every row we obtain

$$\vec{s} = (1 \;\; 0 \;\; 0 \;\; 1 \;\; 1) \tag{3.17}$$

which corresponds to the symmetry

$$(A_1, A_2, A_3, A_4, A_5) \rightarrow (-A_1, A_2, A_3, -A_4, -A_5) \tag{3.18}$$

of the coefficient space. Thus, for example, since we know that in the positive quadrant the shape of the region of stability is the part of the diamond in that quadrant it follows that in the quadrant which corresponds to the above symmetry, the region of stability is again the part of the diamond in that quadrant.

<u>Example 3.3</u>

Consider the class of polynomials above without the last term.

$$P(z_1, z_2, z_3, z_4) = 1 - A_1 z_1 z_2^3 - A_2 z_1 z_2^2 z_3^2 z_4 - A_3 z_1 z_2^3 z_3^3 z_4$$
$$- A_4 z_2^3 z_3^3 z_4 \; . \tag{3.19}$$

Since the rank of J will still be four, and the dimension of the coefficient space is four, it follows that <u>every</u> quadrant is symmetrical to every other by an appropriate simple symmetry. That is, every possible change of sign will occur among the simple symmetries. It follows that the region of stability is the diamond, and

$$|A_1| + |A_2| + |A_3| + |A_4| < 1$$

is a <u>necessary and sufficient</u> condition for the stability of polynomials of this type.

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